

Transience and recurrence of random walks on percolation clusters in an ultrametric space

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Abstract

We study transience and recurrence of simple random walks on percolation clusters in the hierarchical group of order N , which is an ultrametric space. The connection probability on the hierarchical group for two points separated by distance k is of the form $c_k/N^{k(1+\delta)}$, $\delta > 0$, with $c_k = C_0 + C_1 \log k + C_2 k^\alpha$, non-negative constants C_0, C_1, C_2 , and $\alpha > 0$. Percolation occurs for $\delta < 1$, and for the critical case, $\delta = 1$ $\alpha > 0$ and sufficiently large C_2 . We show that in the case $\delta < 1$ the walk is transient, and in the case $\delta = 1, C_2 > 0, \alpha > 0$ there exists a critical $\alpha_c \in (0, \infty)$ such that the walk is recurrent for $\alpha < \alpha_c$ and transient for $\alpha > \alpha_c$. The proofs involve ultrametric random graphs, graph diameters, path lengths, and electric circuit theory. Some comparisons are made with behaviours of simple random walks on long-range percolation clusters in the one-dimensional Euclidean lattice.

Keywords: Percolation, hierarchical group, ultrametric space, random graph, renormalization, random walk, transience, recurrence.

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1 Introduction

Network science is an active field of research due to its many areas of application (statistical physics, biology, computer science, communications, economics, social sciences, etc.), and to the interesting mathematical problems that it gives rise to, many of which remain open. Percolation plays an important role, for example in the study of robustness of networks. Hierarchical networks occur in models where there is a multiscale organization with an ultrametric structure, e.g., in statistical physics (in particular disordered spin systems), protein dynamics, population genetics and computer science. Several areas of physics where ultrametric structures are present were overviewed in [51]. An ultrametric model in population genetics was introduced in [53]. Hierarchical organizations in complex networks were discussed in [6]. A classical ultrametric space is the set of p -adic numbers. A review of many areas where p -adic analysis is used, specially in physics including quantum physics, appeared in [31]. The ultrametric space we deal with in this paper is Ω_N , the *hierarchical group of order N* , described at the end of the Introduction. Background on ultrametric spaces can be found e.g. in [54].

Stochastic models on hierarchical groups have played a fundamental role in mathematical physics and population biology. Dyson [32] introduced such a structure in order to gain insight on the study of ferromagnetic models on the Euclidean lattice of dimension 4, as it provides a “caricature” of the Euclidean lattice in dimensions “infinitesimally close” to 4. A reason for this approach is that it is possible to carry out a renormalization group analysis in a rigorous way in hierarchical groups [10, 17]. Hierarchical groups have also been used in the study of self-avoiding random walks in four dimensions [14], Anderson localization in disordered media [7, 43], mutually catalytic branching in population models [18, 29], interacting diffusions [27, 28], occupation times of branching systems [23, 24], search algorithms [39, 40]. Thus, stochastic models, in particular random walks, on ultrametric spaces are a natural field of study. A class of random walks on hierarchical groups, called *hierarchical random walks*, and their degrees of transience and recurrence were studied in [25, 26] (and references therein). A related model in the context of spin-glass was treated in [49]. Other properties of systems of hierarchical random walks appeared in [11, 12]. An analogous class of hierarchical random walks on the p -adic numbers was studied in [2, 3]. Lévy processes on totally disconnected groups (including the p -adic integers) were discussed in [33], pseudodifferential equations and Markov processes over p -adics were treated in [15]. A random walk model for the dynamics of proteins was discussed in [5]. In this case the states of the walk are related to the local minima of the potential energy of a protein molecule. These are a few representative references on stochastic models on ultrametric spaces.

With these precedents and previous work on percolation in hierarchical groups [21, 22, 42], we were motivated to investigate the behaviour of random walks on percolation clusters in those groups, and to compare results with similar ones for random walks on long-range percolation clusters in Euclidean lattices (referred to below).

The renormalization method for the study of percolation in hierarchical networks involves ultrametric random graphs. An *ultrametric random graph* $URG(M, d)$ is a graph on a finite set of M elements with an ultrametric d and connection probabilities $p_{x,y}$ that are random and depend on the distance $d(x, y)$ (see [22], Section 3.4). A more detailed description related to the model is given in Section 2.

In [21] we studied asymptotic percolation in Ω_N in the limit $N \rightarrow \infty$ (mean field percolation) with a certain class of connection probabilities depending on the distance between points. In this case it was possible to obtain a necessary and sufficient condition for percolation. The Erdős-Rényi theory of giant components of random graphs was a useful tool, although there are significant differences between classical random graphs and ultrametric ones. Percolation in Ω_N with fixed N is technically more involved, and so far only sufficient conditions for percolation or for its absence are known. This was studied in [22], where connectivity results of Erdős-Rényi graphs played a basic role. At the same time an analogous model was studied in [42] using different methods. In [22] the “critical case” was analyzed in more depth. A relationship between the results of [22] and [42] was given in [22] (Remark 3.2). The relevance of percolation in hierarchical groups has been noted for contact processes [4] and epidemiology [34].

In [22] we studied percolation in the hierarchical group (Ω_N, d) , integer $N \geq 2$, ultrametric d , with probability of connection between two points \mathbf{x} and \mathbf{y} such that $d(\mathbf{x}, \mathbf{y}) = k \geq 1$ of the form $p_{\mathbf{x}, \mathbf{y}} = c_k / N^{(1+\delta)k}$, where $\delta > -1$ and the c_k are positive constants, all connections being independent. Here we restrict to $\delta > 0$. The results refer to existence of *percolation clusters* (infinite connected sets) of

positive density. *Percolation* is said to occur if a given point of Ω_N belongs to a percolation cluster with positive probability. The specific point does not matter because the model is translation-invariant. By ultrametricity, percolation is possible only if there exists arbitrarily large k such that $c_k > 0$ (otherwise all connected components are finite). Thus, percolation in Ω_N can be regarded as long-range percolation. Briefly, in [22] the results are: if $\delta < 1$ and $c = \inf_k c_k$ is large enough, then percolation occurs, if $\delta > 1$ and $\sup_k c_k < \infty$, then percolation does not occur, and for the critical case, $\delta = 1$, which is the most delicate, percolation may or may not occur according to some special forms of c_k such that $c_k \rightarrow \infty$ as $k \rightarrow \infty$. When percolation occurs the infinite cluster is unique.

In the critical case c_k was taken of the form

$$(1.1) \quad c_k = C_0 + C_1 \log k + C_2 k^\alpha$$

with non-negative constants C_0, C_1, C_2 , and $\alpha > 0$, and for the case $C_2 > 0$ percolation was established for $\alpha > 2$ and any C_1 if C_0 and C_2 are large enough [22] (Theorem 3.3(a)). The proof was based on a renormalization argument of the type used in statistical physics. Results for the case $C_2 = 0$ were also obtained, in particular if $C_1 < N$, then percolation does not occur for any C_0 . Here we show that percolation occurs for any $\alpha > 0$ if C_2 is large enough (Theorem 2.1), which was an open problem in [22] (section (3.4)). Here we need $\alpha > 0$ for the results on random walks. The proof uses the renormalization ideas introduced in [22], but in a different way which is more intrinsic to the model.

The renormalization approach in [22] was applied for a preliminary percolation result replacing c_k with c'_k of the form

$$(1.2) \quad c'_{k_n} = C + a \log n \cdot n^{b \log N},$$

constants $C \geq 0, a > 0, b > 0$, where

$$(1.3) \quad k_n = \lfloor Kn \log n \rfloor, n = 1, 2, \dots,$$

constant $K > 0$, and $c'_{k_n} \leq c'_k \leq c'_{k_{n+1}}$ for $k_n < k < k_{n+1}$, with some technical conditions on K and b [22] (Theorem 3.5(b)), which was used to prove percolation for $\alpha > 2$. The conditions were $2/\log N < K < b$ (with a minor modification it is possible to have also $K = b$). The proof of percolation for $\alpha > 2$ in [22] is based on the relationship between (1.1) and (1.2) with $\alpha > b \log N$ (proof of Theorem 3.3 in [22]). Note that $c'_k \leq c_k$. We will write c_k for c'_k in (1.2) for simplicity of notation, and no confusion should arise. All one needs to remember regarding (1.1) and (1.2) is $\alpha > b \log N$. Percolation with c'_k implies percolation with c_k as in the proof of Theorem 3.3 in [22]. The scheme with (1.2), (1.3) is not used for the proof of percolation here, but it constitutes a technical tool for the study of behaviour of random walks on percolation clusters regarding some properties of the clusters, hence we will need to refer to some techniques in [22].

The main results in the paper refer to transience and recurrence of simple (nearest neighbour) random walks on the percolation clusters in Ω_N . We show that the random walk is transient for $\delta < 1$ (Theorem 4.4), and in the critical case, $\delta = 1$, $C_2 > 0$, there exists a critical $\alpha_c \in (0, \infty)$ such that the random walk is recurrent for $\alpha < \alpha_c$ and transient for $\alpha > \alpha_c$ (Theorem 4.6).

These results are comparable in part with those on long-range percolation in the one-dimensional Euclidean lattice \mathbb{Z} with connection probabilities of the form $\beta|x - y|^{-s}$ as $|x - y| \rightarrow \infty$, although the Euclidean and the ultrametric structures are quite different. Long-range percolation in \mathbb{Z} with those connection probabilities was introduced by Schulman [55], and studied further for \mathbb{Z}^d by Newman and Schulman [48], and Aizenman and Newman [1]. Berger [9] studied transience and recurrence of random walks on the percolation clusters in \mathbb{Z}^d for $d = 1, 2$. The results for $d = 1$ are, roughly, that percolation can occur if $1 < s \leq 2$, and does not occur if $s > 2$, and if $1 < s < 2$, then the walk is transient, and if $s = 2$, then the walk is recurrent. Hence the results agree for $\Omega_N, 0 < \delta < 1$, and $\mathbb{Z}, 1 < s < 2$, by using the ultrametric $\rho(\mathbf{x}, \mathbf{y}) = N^{d(\mathbf{x}, \mathbf{y})}$ ("Euclidean radial distance") on Ω_N , and $s = \delta + 1$. But there is a significant difference. Percolation in \mathbb{Z} can be obtained by increasing the probability of connection between nearest neighbors (separated by distance 1) [9] (Theorem 1.2), whereas for Ω_N short-range connections play no role due to ultrametricity. Our results for $\delta = 1$ with c_k given by (1.1) would correspond to the case on \mathbb{Z} with $s = 2$ taking β to be a function of distance. Heat kernel bounds and

scaling limits for the walks on the long-range percolation clusters of \mathbb{Z}^d , $d \geq 1$, were obtained in [19, 20]. It would be interesting to find similar results for the hierarchical group.

Grimmett et al [36] studied the behaviour of random walks on (bond) percolation clusters in the Euclidean lattice \mathbb{Z}^d using electric circuit theory [30] (see also [8, 50]). Recurrence of the walk for $d \leq 2$ follows directly from recurrence on the whole space \mathbb{Z}^d , whereas transience for $d \geq 3$ was difficult to prove. For recurrence in long-range percolation the situation is different because the walk is only defined on the percolation cluster (both in \mathbb{Z}^d and in Ω_N). Although the models on \mathbb{Z}^d and Ω_N are quite different, we are able to use some of the basic ideas on the relationship between reversible Markov chains and electric circuits (see e.g. [30, 45, 41]) that have been used for \mathbb{Z}^d , but in the case of Ω_N the ultrametric structure plays a fundamental role.

The transience and recurrence behaviours of walks on the percolation clusters are determined basically by the ultrametric geometry of (Ω_N, d) and the form of the connection probabilities, rather than by detailed properties of the structures of the percolation clusters. The proofs involve some properties of the clusters, in particular cutsets, graph diameters and lengths of paths.

We end the Introduction by recalling (Ω_N, d) and some things about it. For an integer $N \geq 2$, the *hierarchical group* (also called *hierarchical lattice*) of order N is defined as

$$\Omega_N = \{\mathbf{x} = (x_1, x_2, \dots) : x_i \in \mathbb{Z}_N, x_i = 0 \text{ a.a.i.}\}.$$

with addition componentwise mod N , where \mathbb{Z}_N is the cyclic group of order N . The *hierarchical distance* on Ω_N , defined as

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\} & \text{if } \mathbf{x} \neq \mathbf{y}, \end{cases}$$

satisfies the strong (non-Archimedean) triangle inequality,

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{z}), d(\mathbf{z}, \mathbf{y})\} \text{ for any } \mathbf{x}, \mathbf{y}, \mathbf{z}.$$

Hence (Ω_N, d) is an ultrametric space, and it can be represented as the top of an infinite regular N -ary tree where the distance between two points is the number of levels from the top to their closest common node. The point $(0, 0, \dots) \in \Omega_N$ is taken as origin and denoted by $\mathbf{0}$. The probability of an edge connecting two points \mathbf{x} and \mathbf{y} is given by

$$(1.4) \quad p_{\mathbf{x}, \mathbf{y}} = \min\left(\frac{c_k}{N^{(1+\delta)k}}, 1\right) \text{ if } d(\mathbf{x}, \mathbf{y}) = k,$$

where $\delta > 0$ and $c_k > 0$ for every k , all edges being independent. Note that any point in a percolation cluster has a finite (random) number of neighbours since the number has finite expectation, hence the simple (nearest neighbour) random walk on a percolation cluster is well defined.

An essential property of ultrametric spaces that differentiates them from Euclidean spaces is that two balls are either disjoint or one is contained in the other. The following definitions and properties are used throughout. The ball of diameter $k \geq 0$ containing $\mathbf{x} \in \Omega_N$ is defined as $B_k(\mathbf{x}) = \{\mathbf{y} : d(\mathbf{x}, \mathbf{y}) \leq k\}$. Those balls are generally referred to as *k-balls*. They contain N^k points. For $k > 0$, a k -ball is the union of N disjoint $(k-1)$ -balls that are at distance k from each other. For $j > k > 0$, we call $B_j(\mathbf{0}) \setminus B_k(\mathbf{0})$ the *annulus* $(k, j]$, or $(k, j]$ -*annulus*. It contains $N^j(1 - N^{k-j})$ points. A j -ball is the union of N^{j-k} disjoint k -balls. The k -balls in the $(k, j]$ -annulus are at distance at least $k+1$ and at most j from each other. This and (1.4) allow to obtain upper and lower bounds for the probability that subsets of two k -balls in the $(k, j]$ -annulus are connected by at least one edge. Such bounds are used in the proofs.

In Section 2 we prove percolation for $\delta = 1$, $\alpha > 0$. In Section 3 we prepare the tools for the proofs of transience and recurrence on the random walks based on the properties of the clusters. In Section 4 we give the results and proofs of transience and recurrence of the walks using electric circuit theory.

2 Percolation in Ω_N for $\delta = 1$

The results for $\delta < 1$ and $\delta > 1$ have been mentioned in the Introduction. For $\delta = 1$ we regard the model as the infinite random graph

$$\mathcal{G}_N^\infty = \mathcal{G}_N^\infty(C_0, C_1, C_2, \alpha) := \mathcal{G}(V_\infty, \mathcal{E}_\infty)$$

with vertices $V_\infty = \Omega_N$ and edges \mathcal{E}_∞ , and with the probability of connection by an edge (\mathbf{x}, \mathbf{y})

$$(2.1) \quad P((\mathbf{x}, \mathbf{y}) \in \mathcal{E}_\infty) = p_{\mathbf{x}, \mathbf{y}} = \min\left(\frac{c_k}{N^{2k}}, 1\right) \text{ if } d(\mathbf{x}, \mathbf{y}) = k,$$

all connections being independent, and the c_k are of the form

$$(2.2) \quad c_k = C_0 + C_1 \log k + C_2 k^\alpha,$$

with constants $C_2 > 0$ and $\alpha > 0$. For simplicity of notation we set, without loss of generality, $C_0 = C_1 = 0$ in (2.2). The graph \mathcal{G}_N^∞ is a limit of finite graphs of diameter k which we will study as $k \rightarrow \infty$.

Theorem 2.1 *For sufficiently large C_2 , there exists a unique percolation cluster of positive density at least $\varepsilon = \varepsilon(N, C_2, \alpha)$ in Ω_N to which $\mathbf{0}$ belongs with positive probability.*

The proof of this result is given in the next subsection. We begin with the formulation for the renormalization method.

2.1 The hierarchy of random graphs

A collection of vertices in a subset of Ω_N any two of which are linked by a path of edges is called a *cluster* of the subset. We consider for each k -ball a maximal cluster with edges only within the ball and not through paths going outside the ball (i.e., all edges of length $\leq k$ not in the cluster are deleted). If there are more than one (maximal) cluster, then one of them is chosen uniformly at random. In this way each k -ball has a unique attached cluster. The proof will be based on the connections between the clusters in k -balls. When we refer to connections between k -balls we mean direct edge connections (one or more) between their clusters. The *density* of a k -ball is the size of its cluster normalized by the size of the ball (N^k). Due to our assumptions, the densities of different k -balls are i.i.d. An infinite connected subset of Ω_N is called a *percolation cluster*.

The main idea is to consider the distribution of the clusters in the balls $B_k(\mathbf{0})$ of increasing k by relating the random graphs in these balls to a hierarchy of ultrametric random graphs.

2.1.1 Erdős-Rényi graphs with random weights, ultrametric random graphs

In the classical graph $\mathcal{G}(n, p)$ introduced by Gilbert [35] these graphs have a set of n vertices denoted by V and there is an edge between each pair of vertices with probability p with these assigned independently for different pairs (see e.g. [13, 37] for background). The behaviour of these graphs together with the random graphs $\mathcal{G}(n, m)$ in the limit as $n \rightarrow \infty$ were studied by Erdős and Rényi in a series of important papers. We consider a modification of those graphs, namely, $\mathcal{G}(N, \{x_i\}_{i \in V}) = \mathcal{G}(N, \{p(x_i, x_j)\})$ in which the vertices have independent random weights $\{x_i\}_{i \in V}$, and the probability that i and j are connected by an edge is a function $p(x_i, x_j)$, the edges chosen independently conditioned on the weights. These are ultrametric random graphs as stated in the Introduction.

Given \mathcal{G}_N^∞ as above we now introduce a sequence of related finite ultrametric random graphs $\mathcal{G}_k(N, \{X_{k-1}(i)\}_{i \in V_k})$, $k \geq 1$, where $X_0(i) = 1$ and for $k \geq 2$, the $\{X_{k-1}(i)\}_{i \in V_k}$ are the densities of the N disjoint $(k-1)$ -balls in $B_k(\mathbf{0})$ indexed by V_k , $|V_k| = N$. The densities $\{X_{k-1}(i)\}_{i \in V_k}$ are i.i.d. $[0, 1]$ -valued random variables for each k ,

$$(2.3) \quad X_{k-1}(i) = \frac{|\mathcal{C}_{k-1}(i)|}{N^{k-1}}, \quad i \in V_k,$$

where $\mathcal{C}_{k-1}(i)$ denotes the cluster in the i th $(k-1)$ -ball. For N fixed our aim is to determine what happens as $k \rightarrow \infty$. Properties of the graph $\mathcal{G}_k(N, \{X_{k-1}(i)\}_{i \in V_k})$ as $k \rightarrow \infty$ provide information on \mathcal{G}_N^∞ , hence the behaviour of the cluster of $B_k(\mathbf{0})$ as $k \rightarrow \infty$ will imply a result on percolation in Ω_N .

We denote the distribution of $X_k(i)$ by $\mu_k \in \mathcal{P}([0, 1])$. Then we have for each k ,

$$\mu_k = \Phi_k(\mu_{k-1}),$$

where Φ_k is a *renormalization mapping*

$$\Phi_k : \mathcal{P}([0, 1]) \rightarrow \mathcal{P}([0, 1]).$$

Note that μ_k depends on the edges within a $(k-1)$ -ball (which determine μ_{k-1}) and also on the edges between different $(k-1)$ -balls in a k -ball, and that μ_k is an atomic measure.

We now analyse the sequence μ_k , $k \geq 1$, for the class of connection probabilities (2.1), (2.2).

In order to prove percolation of positive density it suffices to construct the sequence of $[0, 1]$ -valued random variables X_k , $k \geq 1$, satisfying

$$P(X_k > a) = \mu_k((a, 1]) \text{ for all } a \in (0, 1),$$

and existence of $a > 0$ such that

$$\liminf_{k \rightarrow \infty} P(X_k > a) > 0.$$

Consider two $(k-1)$ -balls in a k -ball (which are at distance k from each other) having densities x_1, x_2 respectively, and define

$$(2.4) \quad p(x_1, x_2, k) = P(\text{two } (k-1)\text{-balls in a } k\text{-ball with densities } x_1, x_2 \text{ are connected}).$$

Note that $p(x_1, x_2, k)$ is an increasing function of x_1 and x_2 . Then from (2.1), (2.2),

$$(2.5) \quad p(x_1, x_2, k) = 1 - \left(1 - \frac{C_2 k^\alpha}{N^{2k}}\right)^{N^{2(k-1)} x_1 x_2},$$

and

$$(2.6) \quad p(x_1, x_2, k) \sim 1 - e^{-(C_2 k^\alpha / N^2) x_1 x_2} \text{ for large } k.$$

2.1.2 Proof of Theorem 2.1

The idea of the proof is to obtain lower bounds on the expected values of the sequence of random variables $\{X_k(\mathbf{0})\}_{k \geq 1}$, where $X_k(\mathbf{x})$ denotes the *density* of $B_k(\mathbf{x})$, that is $X_k(\mathbf{x}) = |C_k(\mathbf{x})|/N^k$, where $C_k(\mathbf{x})$ denotes the cluster in $B_k(\mathbf{x})$. Then as remarked above μ_k is the probability law of $X_k(\mathbf{x})$ which is independent of \mathbf{x} and the latter will be suppressed. By our assumptions $\{X_k(\mathbf{x}_i)\}$ are independent if for $i \neq j$, $d(\mathbf{x}_i, \mathbf{x}_j) \geq k+1$.

Lemma 2.2 *Let X be a random variable with values in $[0, 1]$ and $0 < a < 1$. Then*

$$(2.7) \quad P(X \geq a/2) \geq \frac{E[X] - a/2}{1 - a/2}.$$

Proof. Let $p = P(X \geq a/2)$. Then

$$E[X] \leq p + \frac{a}{2}(1 - p),$$

hence

$$p \geq \frac{E[X] - a/2}{1 - a/2}.$$

■

Lemma 2.3 *Assume that*

$$(2.8) \quad \liminf_{k \rightarrow \infty} E[X_k] = \liminf_{k \rightarrow \infty} E\left[\frac{|C_k|}{N^k}\right] = a > 0,$$

for some $a > 0$. Then percolation of positive density occurs.

Proof. Lemma 2.2 implies

$$\liminf_{k \rightarrow \infty} P\left(\frac{|C_k|}{N^k} \geq a/2\right) \geq \frac{a}{2-a}.$$

Assume that the expected value of the density of $B_k(\mathbf{0})$ is at least $a/2$ for some $a > 0$ and all large k . Then by transitivity (cf. [22], Lemma 5.6)

$$(2.9) \quad \liminf_{k \rightarrow \infty} P(\mathbf{0} \text{ belongs to the cluster of } B_k(\mathbf{0})) \geq \frac{a^2}{2(2-a)} > 0.$$

On the other hand if we assume that the density of $B_k(\mathbf{0})$ tends to 0 w.p.1, then (2.8) does not hold.

■

To prove Theorem 2.1 we will show that the hypothesis of Lemma 2.3 is satisfied.

We write the sequence of weights X_k , $k \geq 1$, as follows.

$$(2.10) \quad X_{k+1} = \frac{1}{N} \sum_{i=1}^N 1_{i \in C_{k+1}^*} X_{k,i},$$

where $\{X_{k,i}, i = 1, \dots, N\}$ denote the (i.i.d.) densities of the N disjoint k -balls in $B_{k+1}(\mathbf{0})$, and $C_{k+1}^* \subset V_{k+1}$ is the set of indices of the underlying k -balls (some k -balls may have null weights). Hence

$$(2.11) \quad E[X_{k+1}] \leq E[X_k] \text{ for all } k.$$

Now consider the random graph $\mathcal{G}_k(N, \{x_1, \dots, x_N\}) = \mathcal{G}_k(N, \{p(x_i, x_j, k)\})$, where $p(x_i, x_j, k)$ is given by (2.5). Given the densities $(X_{k-1,1}, \dots, X_{k-1,N}) = (x_1, \dots, x_N)$, then the probability that all N $(k-1)$ -balls in a k -ball are connected is

$$(2.12) \quad P(X_k = \frac{1}{N}(x_1 + \dots + x_N) \mid (x_1, \dots, x_N)) = P(\mathcal{G}_k(N, \{p(x_i, x_j, k)\}) \text{ is connected}).$$

If $x_i \geq \varepsilon > 0$ for $i = 1, \dots, N$, then

$$(2.13) \quad P(X_k = \frac{1}{N}(x_1 + \dots + x_N) \mid (x_1, \dots, x_N)) \geq P(\mathcal{G}_k(N, p(\varepsilon, \varepsilon, k)) \text{ is connected}).$$

If all the $(k-1)$ -balls are isolated, then

$$(2.14) \quad P(X_k = \frac{1}{N}(x_1 \vee \dots \vee x_N) \mid (x_1, \dots, x_N)) = \prod_{i < j=1}^N (1 - p(x_i, x_j, k)),$$

where the right side is the probability that no pair of k -balls is connected.

By the independence of the densities of different $(k-1)$ -balls,

$$(2.15) \quad P((X_{k-1,1}, \dots, X_{k-1,N}) = (x_1, \dots, x_N)) = \prod_{i=1}^N \mu_{k-1}(x_i).$$

We now can state a stronger form of (2.8).

Lemma 2.4 For sufficiently large C_2 there exists $a > 0$ such that as $n \rightarrow \infty$,

$$(2.16) \quad E[X_n] \rightarrow a,$$

$$(2.17) \quad \text{Var}[X_n] \rightarrow 0,$$

and

$$(2.18) \quad \mu_n \Rightarrow \delta_a.$$

We first consider the case $N = 2$ to illustrate the idea of the proofs of Lemma 2.4 and Theorem 2.1.

Proof of Lemma 2.4 for $N = 2$. Fix $0 < \varepsilon < 1$ (to be chosen sufficiently small), and let

$$(2.19) \quad \begin{aligned} q_n(\varepsilon) &= \sup\{1 - p(x_1, x_2, n) : x_1, x_2 \geq \varepsilon\} \\ &= P(\text{two } (n-1)\text{-balls in an } n\text{-ball with densities } \geq \varepsilon \text{ are not connected}) \end{aligned}$$

Then $q_n(\varepsilon)$ is decreasing in n for large n , and from (2.5), (2.6),

$$(2.20) \quad 1 - q_n(\varepsilon) \geq 1 - \left(1 - \frac{C_2 n^\alpha}{N^{2n}}\right)^{N^{2(n-1)} x_1 x_2} \sim 1 - e^{-(C_2 n^\alpha / N^2) x_1 x_2} \quad \text{for large } n,$$

$$(2.21) \quad q_n(\varepsilon) \leq (q(\varepsilon))^{n^\alpha},$$

where

$$(2.22) \quad q(\varepsilon) := e^{-C_2 \varepsilon^2 / N^2},$$

hence

$$(2.23) \quad \sum_n q_n(\varepsilon) < \infty.$$

Let

$$(2.24) \quad z_n(\varepsilon) = P(X_n < \varepsilon).$$

By Lemma 2.2,

$$(2.25) \quad r_n(2\varepsilon) := P(X_n \geq 2\varepsilon) \geq \frac{E[X_n] - 2\varepsilon}{1 - 2\varepsilon}.$$

To obtain a lower bound for $E[X_n]$ we first note that

$$(2.26) \quad \begin{aligned} E[X_{n+1} 1_{X_{n+1} \geq \varepsilon}] &\geq \int_0^1 \int_0^1 \frac{x+y}{2} 1_{x+y \geq 2\varepsilon} F_n(dx) F_n(dy) \cdot p(x, y, n) \\ &\geq \int_\varepsilon^1 \int_\varepsilon^1 x F_n(dx) F_n(dy) \cdot (1 - q_n(\varepsilon)) \\ &= (1 - z_n(\varepsilon)) E[X_n 1_{X_n \geq \varepsilon}] \cdot (1 - q_n(\varepsilon)), \end{aligned}$$

where $F_n(dx)$ denotes the distribution of the random variable X_n . Therefore for $n > n_0$ (to be taken sufficiently large),

$$(2.27) \quad \begin{aligned} E[X_n] &\geq E[X_n 1_{X_n \geq \varepsilon}] \\ &= \prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))(1 - q_k(\varepsilon)) E[X_{n_0} 1_{X_{n_0} \geq \varepsilon}]. \end{aligned}$$

From (2.19), (2.24), (2.25),

$$\begin{aligned}
(2.28) \quad z_{n+1}(\varepsilon) &= P(X_{n+1} < \varepsilon) \\
&\leq (P(X_n < \varepsilon))^2 + 2P(X_n < \varepsilon)P(\varepsilon < X_n \leq 2\varepsilon) + (P(\varepsilon < X_n \leq 2\varepsilon))^2 q_n(\varepsilon) \\
&\leq (P(X_n < \varepsilon))^2 + 2P(X_n < \varepsilon)(1 - r_n(2\varepsilon)) + (1 - r_n(2\varepsilon))^2 q_n(\varepsilon) \\
&\leq z_n^2(\varepsilon) + z_n(\varepsilon)2(1 - r_n(2\varepsilon)) + q_n(\varepsilon)(1 - r_n(2\varepsilon))^2 \\
&= z_n(\varepsilon)(z_n(\varepsilon) + 2(1 - r_n(2\varepsilon))) + q_n(\varepsilon)(1 - r_n(2\varepsilon))^2 \\
&\leq z_n(\varepsilon)(z_n(\varepsilon) + 2(1 - r_n(2\varepsilon))) + q_n(\varepsilon)
\end{aligned}$$

where we have used $P(\varepsilon < X_n \leq 2\varepsilon) \leq 1 - r_n(2\varepsilon)$.

In order to prove that $\liminf E[X_n] > 0$ for sufficiently large C_2 , from (2.27) it suffices to verify that we can choose $\varepsilon, n_0, z_{n_0}, E[X_{n_0} 1_{X_{n_0} \geq \varepsilon}]$ and C_2 such that

$$(2.29) \quad \liminf_{n \rightarrow \infty} \prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))(1 - q_k(\varepsilon)) E[X_{n_0} 1_{X_{n_0} \geq \varepsilon}] > 0.$$

Suppose that

$$(2.30) \quad z_{n_0}(\varepsilon) \leq \frac{\varepsilon}{2} \text{ and } C_2 \text{ is large enough so that } q_{n_0}(\varepsilon) < \frac{\varepsilon^2}{4}$$

(see (2.21),(2.22)). Note that C_2 may depend on α (see (2.21)). Assume that

$$(2.31) \quad z_m + 2(1 - r_m(2\varepsilon)) < s, \text{ for } n_0 \leq m \leq n,$$

for some $s \in (0, 1)$ such that

$$(2.32) \quad s + \frac{\varepsilon}{2(1-s)} \leq 1.$$

Then it follows from (2.28),(2.30) that for $n \geq n_0$,

$$(2.33) \quad z_{n+1}(\varepsilon) \leq z_{n_0} s^{n-n_0+1} + \sum_{k=n_0}^n s^{n-k} q_k(\varepsilon) \leq z_{n_0} s^{n-n_0+1} + \frac{q_{n_0}(\varepsilon)}{1-s} \leq \frac{\varepsilon}{2},$$

Then by (2.33), $\{z_n\}_{n_0 \leq m \leq n}$ are bounded by the terms of a summable sequence, namely (see (2.23)),

$$\begin{aligned}
(2.34) \quad \sum_{n=n_0}^{\infty} z_n(\varepsilon, s) &= \frac{\varepsilon}{2} \sum_{n=n_0}^{\infty} s^{n-n_0} + \sum_{n=n_0}^{\infty} \sum_{k=n_0}^n s^{n-k} q_k(\varepsilon) \\
&= \frac{\varepsilon}{2(1-s)} + \sum_{k=n_0}^{\infty} \sum_{n=k}^{\infty} s^{n-k} q_k(\varepsilon) = \frac{\varepsilon}{2(1-s)} + \frac{1}{1-s} \sum_{k=n_0}^{\infty} q_k(\varepsilon) < \infty.
\end{aligned}$$

We first choose $2\varepsilon = 0.1$ and $s = 0.775$ which satisfies (2.32). Then by (2.25), $2(1 - r_n(2\varepsilon)) < 0.75$ and $z_{n_0} + 2(1 - r_n(2\varepsilon)) < s$ provided that $E[X_m] > 2/3$ for $n_0 \leq m \leq n$. We now choose n_0 sufficiently large so that

$$\prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon, s))(1 - q_k(\varepsilon)) > 0.9,$$

and C_2 sufficiently large so that $E[X_{n_0} 1_{X_{n_0} > \varepsilon}] \geq 3/4$ and $z_{n_0} \leq \varepsilon/2$. Note that by choosing C_2 sufficiently large we have $z_{n_0} = 0$ and $E[X_{n_0} 1_{X_{n_0} > \varepsilon}] = 1$, since X_{n_0} is atomic and positive. By continuity, this can also be done for connection probabilities strictly less than 1 but sufficiently close to 1. We then can

verify from (2.27) that $E[X_n] \geq E[X_n 1_{X_n > \varepsilon}] \geq 0.9 \times 3/4 > 2/3$, so we have the consistency condition $E[X_n] > 2/3$ for all $n \geq n_0$, and therefore (2.31) holds for all $n \geq n_0$.

If the two n -balls in the $(n+1)$ -ball have density $\geq \varepsilon$, then by considering the events that the two n -balls are connected, and that they are not connected, it is easy to show that

$$(2.35) \quad E[X_{n+1}] \geq E[X_n](1 - q_n(\varepsilon)) + q_n(\varepsilon) \frac{E[X_n]}{2} = E[X_n](1 - \frac{q_n(\varepsilon)}{2}),$$

which together with (2.11), (2.23) and Lemma 2.3 proves (2.16) ($\{E[X_n]\}$ is a Cauchy sequence), and also

$$(2.36) \quad \text{Var}[X_{n+1}] \leq \frac{1}{2}E[X_n^2] + \frac{1}{2}(E[X_n])^2 - (E[X_n])^2(1 - \frac{q_n(\varepsilon)}{2})^2 \leq \frac{1}{2}\text{Var}[X_n] + q_n(\varepsilon),$$

which proves (2.17) and then (2.35), (2.36) prove (2.18). ■

We now modify the argument with $N = 2$ to prove the theorem for general N . To prepare, we begin with some lemmas.

Lemma 2.5 *Consider the Erdős-Rényi graph $\mathcal{G}(N, 1 - q)$. Let $P(N, q)$ denote the probability that the graph is connected. Then as $q \rightarrow 0$,*

$$(2.37) \quad P(N, q) \geq 1 - C(N)q^{N-1},$$

where $C(N)$ is a constant such that $C(N) \sim N$ for large N

Proof. Recall the basic formula ([35], equation (4), also see [13], p. 198)

$$(2.38) \quad P(N, q) = 1 - \sum_{k=1}^{N-1} \binom{N-1}{k-1} P(k, q) q^{k(N-k)} \geq 1 - \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{k(N-k)}.$$

The result follows by noting that the dominant term in $1 - P(N, q)$ as $q \rightarrow 0$ is given by the smallest power of q on the right side of (2.38) which is $N - 1$ (from the terms $k = 1$ and $k = N - 1$). ■

Corollary 2.6 *Consider the graph $\mathcal{G}_k(N, \{x_1, \dots, x_N\})$. If $x_i \geq \varepsilon$ for all i , then for large k*

$$(2.39) \quad P(\mathcal{G}_k(N, \{x_1, \dots, x_N\}) \text{ is connected}) \geq 1 - q_k^N(\varepsilon),$$

where

$$(2.40) \quad q_k^N(\varepsilon) = C(N)(q_k(\varepsilon))^{N-1}$$

with

$$(2.41) \quad q_k(\varepsilon) = q(N, k, \varepsilon) := e^{-C_2 \varepsilon^2 k^\alpha / N^2} \leq C_3 \gamma^{k^\alpha},$$

with constants $C_3 > 0$ and $0 < \gamma < 1$. Hence

$$(2.42) \quad \sum_k q_k^N(\varepsilon) < \infty.$$

Proof. By (2.20), (2.21), (2.22) the probability that two $(k-1)$ -balls in a k -ball with respective densities $x_1, x_2 \geq \varepsilon$ are connected is given by

$$(2.43) \quad 1 - q_k(\varepsilon) \geq 1 - e^{-C_2 \varepsilon^2 k^\alpha / N^2} \quad \text{for large } k.$$

The result then follows by Lemma 2.5. ■

For the $N(k-1)$ -balls in a k -ball, define

$$(2.44) \quad p(x_1, \dots, x_N, n) = P(\text{all the } (k-1)\text{-balls with densities } x_1, \dots, x_N \text{ are connected}).$$

Then by Corollary 2.6 and analogously as in the case $N = 2$ (see (2.26)),

$$(2.45) \quad \begin{aligned} E[X_{n+1} 1_{X_{n+1} \geq \varepsilon}] &\geq \int_0^1 \dots \int_0^1 \frac{\sum_{i=1}^N x_i}{N} 1_{\sum x_i \geq N\varepsilon} \prod_{i=1}^N F_n(dx_i) \cdot p(x_1, \dots, x_N, n) \\ &\geq \frac{1}{N} \sum_{i=1}^N \int_\varepsilon^1 \dots \int_\varepsilon^1 x_i F_n(dx_i) \prod_{j \neq i} F_n(dx_j) \cdot (1 - q_n^N(\varepsilon)) \\ &= (1 - z_n(\varepsilon))^{N-1} E[X_n 1_{X_n \geq \varepsilon}] \cdot (1 - q_n^N(\varepsilon)), \end{aligned}$$

where

$$(2.46) \quad z_n(\varepsilon) = P(X_n < \varepsilon).$$

Therefore for $n > n_0$,

$$(2.47) \quad E[X_n] \geq E[X_n 1_{X_n \geq \varepsilon}] = \prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))^{N-1} (1 - q_k^N(\varepsilon)) E[X_{n_0} 1_{X_{n_0} \geq \varepsilon}].$$

Let

$$(2.48) \quad r_n(N\varepsilon) = P(X_n \geq N\varepsilon).$$

Then by Lemma 2.2,

$$(2.49) \quad r_n(N\varepsilon) \geq \frac{E[X_n] - N\varepsilon}{1 - N\varepsilon}.$$

Lemma 2.7

$$(2.50) \quad P(X_{n+1} < \varepsilon) \leq NP(X_n < \varepsilon)(1 - r_n(N\varepsilon))^{N-1} + q_n^N(\varepsilon).$$

Proof. We first note that if the density of one of the n -balls in the $(n+1)$ -ball is larger than $N\varepsilon$ then $X_{n+1} > \varepsilon$. Second, if the densities of all the balls are larger than ε and the balls are connected, then $X_{n+1} > \varepsilon$. Therefore

$$\begin{aligned} \{X_{n+1} < \varepsilon\} &\subset \{\text{densities of all } n\text{-balls} < N\varepsilon\} \cap \\ &\quad [\{\text{densities all } n\text{-balls} > \varepsilon \text{ and not connected}\} \cup \{\text{density of at least one } n\text{-ball} < \varepsilon\}]. \end{aligned}$$

Then

$$\begin{aligned} P(X_{n+1} < \varepsilon) &\leq NP(X_n < \varepsilon)(P(X_n < N\varepsilon))^{N-1} \\ &\quad + (P(\varepsilon \leq X_n < N\varepsilon))^N P(\mathcal{G}_n(N, \{\varepsilon, \dots, \varepsilon\}) \text{ not connected}) \\ &\leq NP(X_n < \varepsilon)(P(X_n < N\varepsilon))^{N-1} + (P(\varepsilon \leq X_n < N\varepsilon))^N q_n^N(\varepsilon) \\ &\leq NP(X_n < \varepsilon)(1 - r_n(N\varepsilon))^{N-1} + (1 - r_n(N\varepsilon))^N q_n^N(\varepsilon) \\ &\leq NP(X_n < \varepsilon)(1 - r_n(N\varepsilon))^{N-1} + q_n^N(\varepsilon). \end{aligned}$$

The first summand on the right corresponds to the case that at least one density $< \varepsilon$ and all densities $< N\varepsilon$; the second summand corresponds to the case in which all densities x_i are in $[\varepsilon, N\varepsilon)$ and the balls not connected. ■

From (2.45),(2.49),

$$(2.51) \quad z_{n+1}(\varepsilon) \leq N z_n(\varepsilon)(1 - r_n(N\varepsilon))^{N-1} + q_n^N(\varepsilon).$$

Using (2.43)-(2.51) we proceed analogously as in the case $N = 2$. We can then choose $\varepsilon = 0.1/N$, $E[X_{n_0} 1_{X_{n_0} \geq \varepsilon}] \geq 3/4$, so that (using inequality (2.49)),

$$(2.52) \quad N(1 - r_n(N\varepsilon))^{N-1} \leq 2(1 - r_n(N\varepsilon)) < 0.75,$$

provided that $E[X_n 1_{X_n \geq \varepsilon}] \geq 2/3$. Finally, we can then choose n_0, z_{n_0} and C_2 so that the sequence $\{z_n(\varepsilon)\}$ is summable as in the case $N = 2$ and we have

$$\prod_{k=n_0}^{\infty} (1 - z_k(\varepsilon))^{N-1} (1 - q_k^N(\varepsilon)) > 0.9$$

so that $E[X_n 1_{X_n \geq \varepsilon}] \geq 2/3$ for all $n \geq n_0$ as in the case $N = 2$. ■

Remark 2.8 In the case $\alpha > 1$, combining (2.51), (2.52) with (2.41) we obtain

$$(2.53) \quad z_n(\varepsilon) \leq c \zeta^n$$

with constants $c > 0$, $0 < \zeta < 1$.

Proof of Lemma 2.4 for general N . This follows from (2.11) and the next inequalities which are analogous to (2.35), (2.36), that can be proved similarly to the case $N = 2$ by considering the events that all the N n -balls are connected, or not, and using (2.39), (2.40), (2.41), (2.42):

$$(2.54) \quad E[X_{n+1}] \geq E[X_n] + O(q_n^N(\varepsilon)),$$

and

$$(2.55) \quad \text{Var}[X_{n+1}] \leq \frac{1}{N} \text{Var}[X_n] + O(q_n^N(\varepsilon))$$

as $n \rightarrow \infty$.

The proof of percolation is then finished as in the case $N = 2$ using the previous formulas, and the uniqueness follows from Theorem 1.2 in [42]. This completes the proof of Theorem 2.1. ■

3 Properties of the percolation clusters

In this section we will obtain some properties of the percolation clusters that will be used for studying behaviour or random walks on the clusters. We will use parts of the scheme of [22] referred to in the introduction in the case $\delta = 1$, that is,

$$(3.1) \quad k_n = \lfloor K n \log n \rfloor, \quad n = 1, 2, \dots,$$

$$(3.2) \quad c_{k_n} = C + a \log n \cdot n^{b \log N},$$

$K > 0$, $C \geq 0$, $a > 0$, $b > 0$, $c_{k_n} \leq c_k \leq c_{k_{n+1}}$ for $k_n < k < k_{n+1}$. (3.1) implies that

$$(3.3) \quad k_{n+1} - k_n \sim K \log n \text{ as } n \rightarrow \infty.$$

3.1 Cutsets for $\delta = 1$

A *cutset* of a graph is a set of edges in the graph which, if removed, disconnects the graph.

We consider the percolation cluster of Ω_N in the case $\delta = 1$ with $C_2 > 0$. We will construct a sequence of cutsets for the cluster that will be used to prove recurrence of the random walk on the cluster in the case $\alpha \leq 1$.

The following argument holds with $K = 1$ (in the special case $N = 2$ we need a minor modification which we omit here). First recall that by [22] (Lemma 5.2 with $K = 1$) we have the following result (this does not need the condition $2/\log N < K < b$).

Lemma 3.1 *For $0 < b < 2 - 1/\log N$, with probability one there exists n_0 such that for all $n \geq n_0$ there is no skipping over two successive annuli $(k_n, k_{n+1}]$, that is, there are no single edge connections between the annulus $(k_{n-1}, k_n]$ and the annuli $(k_{n+2}, k_{n+3}]$, $(k_{n+3}, k_{n+4}]$, etc.*

Lemma 3.2 *For any $\alpha > 0$ there exists a sequence of finite cutsets Π_j , $j \geq 1$, for the percolation cluster that are pairwise disjoint for large j , and such that*

$$(3.4) \quad E|\Pi_j| \leq \frac{\kappa_j}{N} \quad \text{for large } j,$$

where

$$(3.5) \quad \kappa_j = a2^\alpha \log j \cdot j^\alpha,$$

with a as in (3.2).

Proof. We take b so that $0 < b < \min(\alpha/\log N, 2 - 1/\log N)$.

Let $I_j = (k_{2j}, k_{2(j+1)}]$ -annulus, then by Lemma 3.1 I_j is connected by edges only to I_{j-1} and I_{j+1} for large j .

Note that for $2(j+1) < \ell \leq 2(j+2)$,

$$c_\ell \lesssim \kappa_j \text{ for large } j,$$

where κ_j is given by (3.5). For a vertex $x \in I_j$, let

$$\mathcal{M}_j(x) = \{\text{vertices in } I_{j+1} \text{ connected to } x \text{ by an edge}\}.$$

Then

$$|\mathcal{M}_j(x)| = \sum_{\ell=k_{2(j+1)}+1}^{k_{2(j+2)}} \text{Bin}(N^\ell - N^{\ell-1}, c_\ell/N^{2\ell}).$$

By ultrametricity, the distribution of $|\mathcal{M}_j(x)|$ is the same for any $x \in I_j$. Then by (3.3)

$$\begin{aligned} E|\mathcal{M}_j(x)| &= \left(1 - \frac{1}{N}\right) \sum_{\ell=k_{2(j+1)}+1}^{k_{2(j+2)}} \frac{c_\ell}{N^\ell} \\ &\sim \kappa_j \left(\frac{1}{N^{k_{2(j+1)}+1}} - \frac{1}{N^{k_{2(j+2)}+1}} \right) \\ &\sim \frac{\kappa_j}{N N^{k_{2(j+1)}}} \text{ for large } j. \end{aligned}$$

Let

$$\Pi_j = \{\text{edges connecting vertices in } I_j \text{ restricted to the cluster and vertices in } I_{j+1}\}.$$

Then the sets Π_j are finite cutsets for the cluster and they are pairwise disjoint for large j . Hence

$$E|\Pi_j| \leq |I_j| E|\mathcal{M}_j| \lesssim N^{k_{2(j+1)}} \frac{\kappa_j}{N N^{k_{2(j+1)}}} \lesssim \frac{\kappa_j}{N} \text{ for large } j.$$

■

3.2 Graph diameters and path lengths

In this part we will obtain bounds for the lengths of paths in the percolation clusters joining two points within distance k_n from $\mathbf{0}$ for large n , for $\delta = 1$ and $\delta < 1$, which are of independent interest. This will be done by means of known results on diameters of random graphs [16, 52]. However, for the proof of transience of the random walk on the cluster in the case $\delta = 1$ we will need a stronger result with a probability bound.

We assume that n_0 is large enough according to the proofs of Theorems 3.1(b) and 3.5(b) in [22] (we will refer to parts of those proofs). This means that the things we will do are possible for $n \geq n_0$, in particular there exist the direct edge connections between clusters we will refer to. If the two points are in the same k_{n_0} -cluster, the length of a path joining them is bounded by the diameter of the cluster. Therefore we will assume that the two points lie in the clusters of different k_{n_0} -balls. We proceed as follows:

- Find bounds for the diameters of the Erdős-Rényi random graphs $G(\mathcal{N}_n, p_n)$ defined below, whose vertices are k_n -balls in a k_{n+1} -ball, and the connection probability p_n is defined in terms of direct edges between the clusters of those k_n -balls. These graphs are also ultrametric random graphs.
- Since the k_n -balls consist of k_{n-1} -balls, find bounds for the length of a path of k_{n-1} -balls in a k_n -ball connecting an incoming k_{n-1} -ball and an outgoing k_{n-1} -ball (this may be called a k_n -level path). Such a path may visit a k_{n-1} -ball more than one time, but that does not matter because we consider shortest paths.
- Having done the previous two things, do the same going from n to $n - 1$, etc., down to n_0 , where we end up with the diameters of the clusters of k_{n_0} -balls which are i.i.d., and we denote their expected value by $D(n_0)$.

In the arguments and calculations for path lengths we may think of paths within the ball $B_{k_n}(\mathbf{0})$ joining $\mathbf{0}$ to a point in the $(k_{n-1}, k_n]$ -annulus. However, by ultrametricity any point in $B_{k_n}(\mathbf{0})$ is a center, so the bounds hold as well for paths joining any two points within distance k_n from $\mathbf{0}$.

We recall from [22] (Def. 4.1, Def. 5.5) that a k_n -ball is “good” if its cluster has size at least $N^{\gamma k_n}$ for $\delta < 1$, where $(1 + \delta)/2 < \gamma < 1$, and if its cluster has size at least βN^{k_n} for $\delta = 1$, with some $0 < \beta < 1$. In the case $\delta = 1$ we assume $b > K > 2/\log N$, which corresponds to $\alpha > 2$. Under these conditions, in the proofs of Theorems 3.1(b) and 3.5(b) it is shown that for all but finitely many n the k_n -balls in any increasing nested sequence are good (see [22], (4.22) for $\delta < 1$, (5.11), (5.23) for $\delta = 1$).

Let \mathcal{N}_n denote the number of good k_n -balls in a k_{n+1} -ball, and

$$(3.6) \quad p_n = P(\text{the clusters in two good } k_n\text{-balls in a } k_{n+1}\text{-ball are connected}).$$

Note that p_n is random because the sizes of the clusters are random. We consider the Erdős-Rényi graphs $G(\mathcal{N}_n, p_n)$. In all the cases, $\mathcal{N}_n \rightarrow \infty$ as $n \rightarrow \infty$. In the proof of Theorem 5.3(b) in [22] it is shown that for $\delta = 1$, $b \geq 1$, and $\beta > 1/5$, the graph $G(\mathcal{N}_n, p_n)$ becomes connected for large n . We shall see that for $b > 2$ it even becomes complete (all pairs of vertices are connected).

3.2.1 Diameters of the graphs $G(\mathcal{N}_n, p_n)$

First we obtain bounds for the diameters of the graphs $G(\mathcal{N}_n, p_n)$ in the following cases where except in case 2 we assume that $K = 1$.

Case 1. $\delta < 1$.

From [22] ((4.5), (4.8), (4.10)), we have the lower bound for (3.6)

$$p_n \geq 1 - \exp(-cN^{2\gamma k_n - (1+\delta)k_{n+1}}) > 1 - \exp(-cN^{\varepsilon n \log n}) \quad \text{as } n \rightarrow \infty,$$

with some $0 < \varepsilon < 1$, hence $p_n \rightarrow 1$ and $\mathcal{N}_n p_n / \log \mathcal{N}_n \rightarrow \infty$ as $n \rightarrow \infty$, therefore by Theorem 2 in [16] $\text{diam}(G(\mathcal{N}_n, p_n))$ is concentrated on at most two values $\{1, 2\}$ at

$$\frac{\log \mathcal{N}_n}{\log(\mathcal{N}_n p_n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

which implies that $\text{diam}(G(\mathcal{N}_n, p_n)) \leq 2$ for large n .

Case 2. $\delta = 1$, $b > 2K$.

Lemma 3.3 Assume that $b > 2K$. Then the graph $G(\mathcal{N}_n, p_n)$ is complete and $\text{diam}(G(\mathcal{N}_n, p_n)) = 1$ for large n .

Proof. From [22] (proof of Lemma 5.7 except the last step), we have

$$p_n \geq 1 - \exp(-\beta^2 a \log n \cdot N^{(b-2K) \log n}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Then, since $\mathcal{N}_n \leq N^{K \log n}$,

$$\begin{aligned} q_n &:= P(\text{some pair of clusters of two good } k_n\text{-balls in a } k_{n+1}\text{-ball is not connected}) \\ &\leq N^{2K \log n} (1 - p_n) \leq N^{2K \log n} \exp(-\beta^2 a \log n \cdot N^{(b-2K) \log n}) \\ &= n^{2K \log N} / n^{\beta^2 a n^{(b-2K) \log N}}, \end{aligned}$$

Therefore $\sum q_n < \infty$, hence by Borel-Cantelli for all but finitely many n the graph $G(\mathcal{N}_n, p_n)$ is complete. So, $\text{diam}(G(\mathcal{N}_n, p_n)) = 1$ for large n . ■

Case 3. $\delta = 1, b = 2$.

As in case 2,

$$p_n \geq 1 - \exp(-\beta^2 a \log n) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

then as in case 1, $\text{diam}(G(\mathcal{N}_n, p_n)) \leq 2$ for large n .

Case 4. $\delta = 1, 1 < b < 2$.

Again as above,

$$p_n \geq 1 - \exp\left(-\frac{\beta^2 a \log n}{N^{(2-b) \log n}}\right) > \frac{\beta^2 a \log n}{2N^{(2-b) \log n}}$$

for large n such that

$$\frac{\beta^2 a \log n}{2N^{(2-b) \log n}} < 0.7968.$$

Since $\mathcal{N}_n \leq N^{\log n}$, and from the proof of Theorem 3.5(b) in [22] ((5.17), (5.18) and step 1) we have $\mathcal{N}_n \geq \beta N^{\log n}$ for large n , then

$$\frac{\mathcal{N}_n p_n}{\log \mathcal{N}_n} \geq \frac{\beta^3 a \log n \cdot N^{b \log n}}{2N^{\log n} \log n \cdot \log N} = \frac{\beta^3 a}{2 \log N} N^{(b-1) \log n} \rightarrow \infty \text{ as } n \rightarrow \infty,$$

then, again by Theorem 2 in [16], $\text{diam}(G(\mathcal{N}_n, p_n))$ is concentrated on at most two values at

$$\begin{aligned} \frac{\log \mathcal{N}_n}{\log(\mathcal{N}_n p_n)} &\leq \frac{\log n \cdot \log N}{\log\left(\frac{\beta^3 a}{2} \log n \cdot N^{(b-1) \log n}\right)} \\ &\sim \frac{\log n \cdot \log N}{\log(\log n \cdot N^{(b-1) \log n})} \lesssim \frac{1}{b-1} \text{ as } n \rightarrow \infty, \end{aligned}$$

so, $\text{diam}(G(\mathcal{N}_n, p_n)) \leq b/(b-1)$ for large n . Note that this bound is continuous at $b = 2$ (case 3).

Case 5. $\delta = 1, b = 1$.

By Theorem 1.2 of [52] and [22] (Lemma 5.7),

$$p_n \gtrsim \frac{\beta^2 a \log n}{N^{\log n}} = \frac{\lambda_n}{\mathcal{N}_n} =: \tilde{p}_n \text{ for large } n,$$

where

$$\lambda_n = \beta^2 a \log n \cdot \frac{\mathcal{N}_n}{N^{\log n}} \leq \beta^2 a \log n \leq \mathcal{N}_n^{1/1000} \text{ for large } n,$$

hence $\text{diam}(G(\mathcal{N}_n, \tilde{p}_n))$ is concentrated on two values around

$$f(\mathcal{N}_n, \lambda_n) = \frac{\log \mathcal{N}_n}{\log \lambda_n} + \frac{2 \log \mathcal{N}_n}{\log(1/\lambda_n^*)} + O(1) \text{ for large } n,$$

where

$$\lambda_n^* e^{-\lambda_n^*} = \lambda_n e^{-\lambda_n}, \quad \lambda_n \rightarrow \infty, \quad \lambda_n^* \rightarrow 0,$$

then

$$\begin{aligned} \log \lambda_n &\sim \log \log n, \\ \log \lambda_n^* - \lambda_n^* &= \log \lambda_n - \lambda_n, \end{aligned}$$

since $\mathcal{N}_n \geq \beta N^{\log n}$ for large n , then

$$\log(1/\lambda_n^*) = \lambda_n - \log \lambda_n - \lambda_n^* \gtrsim \beta^3 a \log n,$$

and

$$f(\mathcal{N}_m, \lambda_n) \lesssim \frac{\log \mathcal{N}_n}{\log \log n} + C_1 \frac{\log \mathcal{N}_n}{\log n} + O(1) \lesssim C_2 \log n \quad \text{for large } n,$$

so, $\text{diam}(G(\mathcal{N}_n, p_n)) \lesssim L \log n$ for some constant $L > 0$ and large n .

3.2.2 Path lengths in \mathcal{G}_N^∞

We will now obtain a bound $L(n)$ for the expected length of a path joining two points in a k_n -ball for large n (recall that this is the same as the expected length of a path from $\mathbf{0}$ to a point in the $(k_{n-1}, k_n]$ -annulus). This follows the three steps in the procedure described at the beginning of subsection 3.2, once we have results on the expected values of the diameters on the graphs $G(\mathcal{N}_n, p_n)$. We use here expected values of the diameters of the graphs for a general argument, but we actually found bounds for the diameters themselves for large n in the five special cases considered above. Note however that we do the calculation by an iteration in reverse order, that is, starting with $L(n_0) = D(n_0)$ and ending with $L(n_0 + j)$.

Let n_0 be large enough as mentioned above. We have denoted by $D(n_0)$ the expected diameter of the cluster in a k_{n_0} -ball. Let $D(n_0 + j) = E[\text{diam}(G(\mathcal{N}_{n_0+j}, p_{n_0+j}))]$, $j \geq 1$, $L(n_0 + j)$ denote a bound for the expected length of a path joining $\mathbf{0}$ to a point in the $(k_{n_0+j-1}, k_{n_0+j}]$ -annulus, $j \geq 1$, $L(n_0) = D(n_0)$.

Then $L(n_0 + 1) = D(n_0)(D(n_0 + 1) + 1) + D(n_0 + 1)$, because there are at most $D(n_0 + 1)$ edges in the k_{n_0+1} -ball that join $D(n_0 + 1) + 1$ k_{n_0} -balls, considering the two ends of the k_{n_0+1} -level path (a path of k_{n_0} -clusters), and that the path may enter and leave each k_{n_0} -cluster from different points that are joined by a path of length at most $D(n_0)$ in the k_{n_0} -cluster. So,

$$L(n_0 + 1) = D(n_0)D(n_0 + 1) + D(n_0) + D(n_0 + 1).$$

Similarly,

$$\begin{aligned} L(n_0 + 2) &= L(n_0 + 1)(D(n_0 + 2) + 1) + D(n_0 + 2) \\ &= D(n_0)D(n_0 + 1)D(n_0 + 2) + D(n_0)D(n_0 + 2) + D(n_0 + 1)D(n_0 + 2) \\ &\quad + D(n_0)D(n_0 + 1) + D(n_0) + D(n_0 + 1) + D(n_0 + 2) \\ &= (D(n_0) + 1)(D(n_0 + 1) + 1)(D(n_0 + 2) + 1) - 1, \end{aligned}$$

We show that

$$L(n_0 + k) = \prod_{j=0}^k (D(n_0 + j) + 1) - 1 \quad \text{for } k \geq 0$$

by induction:

$$\begin{aligned} L(n_0 + k + 1) &= L(n_0 + k)(D(n_0 + k + 1) + 1) + D(n_0 + k + 1) \\ &= \left(\prod_{j=0}^k (D(n_0 + j) + 1) - 1 \right) (D(n_0 + k + 1) + 1) + D(n_0 + k + 1) \\ &= \prod_{j=0}^{k+1} (D(n_0 + j) + 1) - 1. \end{aligned}$$

Hence

$$L(n_0 + j) \leq \prod_{k=0}^j (D(n_0 + k) + 1), \quad j \geq 1.$$

It follows that if $D(n_0 + k) < L$ for $k > 1$, with $L > 1$ (cases 1, 3 and 4 above), then

$$(3.7) \quad L(n_0 + j) < (D(n_0) + 1)(L + 1)^j \quad \text{for } j \geq 1.$$

and if $D(n_0 + k) < L_1 \log k$ for $k > 1$, with $L_1 > 0$ (case 5 above) then

$$L(n_0 + j) < (D(n_0) + 1)(L \log j)^j \quad \text{for } j \geq 1 \text{ and some constant } L > 0.$$

The results that will be used for the proofs below for the cases $\delta < 1$, and $\delta = 1, b > 2$ are:

Lemma 3.4 *In the case $\delta < 1$,*

$$(3.8) \quad L(n_0 + j) \leq K(n_0)(L + 1)^j \quad \text{for } j \geq 1,$$

where $K(n_0) = D(n_0) + 1$.

Proof. This follows from case 1 and (3.7). ■

This result could be made stronger by an argument of the type in the next Lemma, but it suffices as it is to prove transience of the random walk on the percolation cluster for $\delta < 1$.

Lemma 3.5 *In the case $\delta = 1, b > 2K > 2$,*

$$(3.9) \quad L(n_0 + j) \leq CD(n_0)j \quad \text{for } j \geq 1 \quad \text{and some constant } C > 0.$$

Proof. Here we also refer to the setup of Section 2. To obtain an upper bound on the expected path length between two points \mathbf{x}, \mathbf{y} in a ball of diameter k_{n_0+j} we find a path with edges with hierarchical lengths $k_{n_0+j}, k_{n_0+j-1}, \dots, k_{n_0+1}$ corresponding to connections between disjoint k_{n_0+j-1} -balls, k_{n_0+j-2} -balls, \dots , k_{n_0+1} -balls respectively, and paths within k_{n_0} -balls. We first calculate an upper bound on the expected number of edges of length k_{n_0+j} needed. If the points are in the same k_{n_0+j-1} -ball this is 0, if they are in connected k_{n_0+j-1} -balls this is 1 and otherwise the length of the path is bounded by $N^{K \log n}$. The probability that two k_{n_0+j-1} -balls having densities at least ε are connected is given by

$$\begin{aligned} & P(\text{two } k_n\text{-balls with densities } \geq \varepsilon \text{ in } B_{k_{n+1}}(\mathbf{0}) \text{ are connected}) \\ & \gtrsim 1 - \left(1 - \frac{a \log n \cdot N^{b \log n}}{N^{2k_{n+1}}}\right)^{\varepsilon^2 N^{2k_n}} \\ & \sim 1 - \exp\left(-a\varepsilon^2 \log n \cdot N^{b \log n - 2(k_{n+1} - k_n)}\right) \\ & \sim 1 - \exp\left(-a\varepsilon^2 \log n \cdot n^{(b-2K) \log N}\right). \end{aligned}$$

Therefore for $\alpha > 1$, recalling (2.46),

$$\begin{aligned} s_n &:= P(\text{two } k_n\text{-balls in } B_{k_{n+1}}(\mathbf{0}) \text{ are connected}) \\ & \gtrsim 1 - \exp\left(-a\varepsilon^2 \log n \cdot n^{(b-2K) \log N}\right) - 2z_{k_n}(\varepsilon) \\ & \sim 1 - \exp\left(-a\varepsilon^2 \log n \cdot n^{(b-2K) \log N}\right) - c\zeta^{Kn \log n}. \end{aligned}$$

for some constant $c > 0$ and $0 < \zeta < 1$ (using (2.53) with k_n in the place of n). Therefore the expected number of edges of length k_{n_0+j} is bounded by

$$e_{n_0+j} = 1 + N^{K \log(n_0+j)} [\exp\left(-a\varepsilon^2 \log(n_0+j) \cdot (n_0+j)^{(b-2K) \log N}\right) + c\zeta^{K(n_0+j) \log(n_0+j)}].$$

We must then connect the entrance vertices and the exit vertices in each of the k_{n_0+j-1} -balls and following the same procedure the bound for the expected number of length k_{n_0+j-1} edges needed in a given k_{n_0+j-1} -ball is given by e_{n_0+j-1} , and since the random variables involved are independent we obtain that the expectation of the total number of length k_{n_0+j-1} edges needed is bounded by $e_{n_0+j} \cdot e_{n_0+j-1}$. Continuing we obtain that the upper bound for the expected number of edges of length k_{n_0+1} is

$$\prod_{\ell=1}^j e_{n_0+\ell}.$$

Noting that since $b - 2K > 0$ and $0 < \zeta < 1$,

$$\lim_{j \rightarrow \infty} \prod_{\ell=1}^j e_{n_0+\ell} = \prod_{\ell=1}^{\infty} \left(1 + N^{K \log(n_0+\ell)} \left[\exp \left(-a\varepsilon^2 \log(n_0+1) \cdot (n_0+\ell)^{(b-2K) \log N} \right) + c\zeta^{K(n_0+\ell) \log(n_0+\ell)} \right] \right) < \infty,$$

and that the expected path length between points \mathbf{x} and \mathbf{y} is bounded by

$$\sum_{m=1}^j \prod_{\ell=1}^m e_{n_0+\ell} D(n_0) \leq \left(\prod_{\ell=1}^{\infty} e_{n_0+\ell} \right) D(n_0) j$$

where $D(n_0)$ is the expected path length between points in a k_{n_0} -ball, the proof is finished. \blacksquare

4 Random walks on the percolation cluster

4.1 Random walks and electric circuits

In this subsection we review briefly some basic background on random walks and electric circuits on graphs which will then be applied to random walks on the percolation clusters.

The nearest neighbour random walk on a finite or an infinite graph such as the percolation cluster is a Markov chain on the countable connected subset given by the graph. Here there is a transition between neighbours x and y with probability

$$p_{xy} = \frac{1}{n(x)},$$

where $n(x)$ is the number of neighbours of x in the graph.

The random walk is a *reversible* since setting $\pi(x) = n(x)$ we have

$$C(x, y) = \pi(x)p_{xy} = \pi(y)p_{yx} \text{ for all } x, y.$$

In this case $C(x, y)$ is called the *conductance* between x and y and the *resistance* $R(x, y)$ is defined as $R(x, y) = 1/C(x, y)$.

For any finite set Z of vertices the *effective conductance* and *effective resistance* between a point a and Z are defined as

$$\mathcal{C}(a \leftrightarrow Z) = \pi(a)P(\tau_Z < \tau_a^+), \quad \mathcal{R}(a \leftrightarrow Z) = 1/\mathcal{C}(a \leftrightarrow Z).$$

where τ_a^+ is the first time after 0 that walk visits a and τ_Z is the hitting time of Z .

If G is an infinite connected graph, let G_n be a finite subgraph of G such that $G_n \uparrow G$ as $n \rightarrow \infty$ and $Z_n := G \setminus G_n$ (identified as a single vertex). Then the *effective resistance from a to ∞* is defined as

$$\mathcal{R}(a \leftrightarrow \infty) = \lim_{n \rightarrow \infty} \frac{1}{\mathcal{C}(a \leftrightarrow Z_n)}.$$

4.2 Criteria for transience and recurrence

Doyle and Snell [30] (also see [45, 41]) proved that the effective resistance is equivalent to the resistance computed using the laws of electric circuit theory applied to the circuit obtained by replacing each edge by a unit resistor resulting in the following criterion for transience and recurrence.

Criterion for transience-recurrence The random walk on an infinite connected graph is transient, respectively recurrent, if $\mathcal{R}(a \leftrightarrow \infty)$ is finite, respectively infinite.

Rayleigh monotonicity principle Removing an edge increases the resistance between two points. Therefore to prove that the random walk on the graph is transient it suffices to show that it is transient on a subgraph.

We will use a related criterion for transience based on the Dirichlet's minimization principle for energy of a flow in a circuit.

Definition 4.1 A unit flow on an infinite graph $G = (V, E)$ with source $a \in V$ is a function θ on the set of edges E such that $\theta(x, y) = -\theta(y, x)$ and for all $x \neq a$,

$$\sum_{x \neq a} \theta(a, x) = 1 \quad \text{and} \quad \sum_{y \sim x} \theta(x, y) = 0 \text{ for all } x \neq a,$$

where $x \sim y$ means that y is a neighbour of x .

Definition 4.2 The energy of the flow is

$$\mathcal{E}(\theta) = \sum_{e \in E^*} (\theta(e))^2 R(e),$$

where E^* is the set of directed edges and $R(e) := R(x, y)$ is the resistance of the edge e from x to y .

4.2.1 Transience, finite energy criterion

A random walk on a countable connected graph G is transient iff there is a unit flow from any vertex a to ∞ on G with finite energy [44].

4.2.2 Recurrence, Nash-Williams criterion

If $\{\Pi_n\}$ is a sequence of disjoint finite cutsets in a locally finite graph G , each of which separates a from infinity, then

$$\mathcal{R}(a \longleftrightarrow \infty) \geq \sum_n \left(\sum_{e \in \Pi_n} C(e) \right)^{-1}.$$

In particular, if the right-hand side is infinite, then the walk on G is recurrent [47]. In our case the edges have unit resistance, so the random walk is recurrent if

$$\sum_n 1/|\Pi_n| = \infty.$$

4.3 Transience and recurrence of random walks on the percolation cluster

In this subsection we give transience and recurrence results for simple (nearest neighbour) random walks on the percolation clusters for $\delta < 1$ and for $\delta = 1$, $C_2 > 0$.

4.3.1 Recurrence for $\delta = 1, \alpha \leq 1$

Theorem 4.3 *In the case $\delta = 1$, sufficiently large C_2 , for almost every realization of the percolation cluster the random walk on the cluster is recurrent if $\alpha \leq 1$.*

Proof. For the cutsets Π_j in Lemma 3.2 (note that for $\alpha \leq 1$ and $N \geq 2$, $\alpha/\log N \leq 2 - 1/\log N$, see choice of b at the beginning of the proof of Lemma 3.2).

$$E\left(\frac{1}{|\Pi_j|}\right) \geq \frac{1}{E|\Pi_j|},$$

(by Jensen's inequality since $1/x$ is convex on $(0, \infty)$), hence by (3.4)

$$E\left(\frac{1}{|\Pi_j|}\right) \gtrsim \frac{N}{\kappa_j} \text{ for large } j.$$

Then by (3.5)

$$E\left(\sum_j \frac{1}{|\Pi_j|}\right) \geq N \sum_j \frac{1}{\kappa_j} = \infty,$$

since $\alpha \leq 1$.

The random variables $1/|\Pi_j|$ are independent and bounded by 1, hence the probability that $\sum_j 1/|\Pi_j|$ diverges is positive ([38], Prop. 4.14), then by Kolmogorov's 0-1 law $\sum_j 1/|\Pi_j|$ diverges w.p.1. Then the recurrence of the random walk follows by the Nash-Williams criterion. ■

4.3.2 Transience for $\delta < 1$

Theorem 4.4 *In the case $\delta < 1$, for almost every realization of the percolation cluster the random walk on the cluster is transient.*

Proof. Let $k_n = \lfloor n \log n \rfloor$, $c = \inf_k c_k > 0$, $A_n = (k_{n-1}, k_n]$ -annulus, and denote by \mathcal{M}_n the number of edges connecting A_n and A_{n+1} . Then \mathcal{M}_n stochastically dominates

$$B_n = \text{Bin}\left(|A_n||A_{n+1}|, \frac{c}{N^{(1+\delta)k_{n+1}}}\right),$$

hence

$$E\mathcal{M}_n \gtrsim N^{2k_n} \frac{c}{N^{(1+\delta)k_{n+1}}} \sim cN^{(1-\delta)n \log n} \text{ as } n \rightarrow \infty.$$

Since $cN^{(1-\delta)n \log n} \gg 4^n$ as $n \rightarrow \infty$, and $\text{Var}[B_n] = O(E[B_n])$ as $n \rightarrow \infty$, it can be shown using [38] (Lemma 4.1) that

$$(4.1) \quad P(\mathcal{M}_n > 4^n) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Therefore by (4.1) for all large n we can pick 4^n direct edges from A_{n-1} to A_n . Since $|A_n| \geq (1 - \varepsilon)N^{n \log n}$ for some $0 < \varepsilon < 1$, we can subdivide A_n into $4^n + 4^{n+1}$ disjoint subsets, each containing $O(N^{n \log n - \log 4 / \log N})$ vertices. We assign 4^n of these subsets as *entrance-sets* for edges from A_{n-1} , and 4^{n+1} of them as *exit-sets* for edges to A_{n+1} , and identify an edge from each one of the 4^n exit-sets in A_{n-1} to a different entrance set in A_n . The end-points of those edges are an *in-vertex* in the entrance-set, and an *out-vertex* in the exit-set in the previous annulus.

We now connect by an edge each one of the $4^n \cdot 4^{n+1}$ pairs (entrance-set, exit-set) in A_n . The probability that there is no such pair connection is

$$\sim \left(1 - \frac{c}{N^{(1+\delta)n \log n}}\right)^{N^{2n \log n}} \sim \exp(-cN^{(1-\delta)n \log n}) \text{ as } n \rightarrow \infty,$$

hence the probability of the event that any of the pairs fail to be connected is

$$O(4^{2n} \exp(-cN^{(1-\delta)n \log n})) \text{ as } n \rightarrow \infty.$$

Since this is summable, then by Borel-Cantelli there exists a random n_0 such that for all $n \geq n_0$ all the pairs in A_n are connected. Therefore we can construct an infinite tree which is rooted at some vertex in A_{n_0} , whose nodes are in-vertices in entrance-sets in successive annuli A_n , such that each node has 4 children, one in each of 4 different entrance-sets in the next annulus (all disjoint), which are connected by edges to corresponding 4 different exit-sets in the previous annulus, so that the n -th generation consists of 4^n vertices.

It remains to connect by paths within each entrance-set its in-vertex to the end-points of the edges that connect the entrance-set to its corresponding 4 exit-sets in the annulus (these end-points are *out-vertices* in the entrance set), and similarly to connect by paths within each exit-set its out-vertex to the end-point of the edge from its corresponding entrance-set (this end-point is an *in-vertex* in the exit-set). In order for such paths to exist we need to assume that all this is done within the percolation cluster. Since the clusters in good k_n -balls have size at least $N^{\gamma k_n}$ with $(1+\delta)/2 < \gamma < 1$ for all sufficiently large n (see [22], Def. 4.1, (4.4), (4.6), (4.22)), we take N^γ instead of N above, so that the construction takes place in the percolation cluster. Then the connecting paths exist and there may be more than one in each set. For $n > n_0$, the length of a path from a node of the tree in A_n to any one of its children in the next generation is bounded by $1 + 1 + K(n_0)(3^{n-n_0} + 3^{n-n_0})$, with $K(n_0)$ given in Lemma 3.4. The 1's come from the single edges between an annulus A_n and the next one, and from the single edges connecting out-vertices in entrance-sets to in-vertices in exit-sets in the annulus. The $3^{(n-n_0)}$'s come from the lengths of paths joining the in-vertex to the out-vertices in an entrance set, and the length of a path joining the in-vertex and the out-vertex in an exit-set in A_n . Hence the length of a path from a node in the tree in the n th generation to any of its 4 children in the next generation is bounded by $C3^n$, by Lemma 3.4 for some positive constant C .

By construction the paths joining a node in the tree to its children in the next generation can have common edges only within the entrance-set. Since there are at least 1 and at most 4 out-vertices in an entrance-set, then each edge in the paths is used at most 4 times. Then it follows by Proposition 3 of [36], with $\beta = 3$ and $\alpha = \gamma = 4$, that the resistance of the tree from the root to infinity is at most

$$4 \sum_{n=n_0}^{\infty} \frac{C3^n}{4^n} < \infty.$$

Therefore, by the criterion for transience-recurrence in subsection 4.2 the walk on the percolation cluster is transient. ■

4.3.3 Transience for $\delta = 1$, $\alpha > 6$.

Theorem 4.5 *In the case $\delta = 1$, sufficiently large C_2 and $\alpha > 6$, for almost every realization of the percolation cluster the random walk on the cluster is transient.*

Proof. The idea of the proof is to construct a subgraph of the percolation cluster \mathcal{C} and a flow on the subgraph that satisfies the finite energy condition criterion.

We assume that k_n is given as in (3.1) with $2/\log N < K < b < \alpha/\log N$, and in this proof we take $b > 3K$. Hence we have the condition of Lemma 3.3.

Given $\alpha > 1$ and sufficiently large C_2 , there exists n_{00} such that for all $n \geq n_{00}$ all the $N^{K \log n}$ k_n -balls in $B_{k_{n+1}}(\mathbf{0})$ are β -good with $\beta = \varepsilon$ where ε is as in the proof of Theorem 2.1. This follows since the probability that a k_n -ball is not ε -good (i.e. $X_{k_n} < \varepsilon$) is $< c\zeta^{K n \log n}$ (see (2.46), (2.53)), and

$$\sum_n N^{K \log n} \cdot \zeta^{K n \log n} < \infty.$$

The edges of the subgraph will be decomposed into a sequence of subsets:

- edges connecting successive $A_n = (k_n, k_{n+1}]$ -annulus, $n = 1, 2, \dots$, $k_n = \lfloor K n \log n \rfloor$,

- these edges go from disjoint good k_n -balls in A_n to disjoint good k_{n+1} -balls in A_{n+1} ,
- there are also edges within the k_n -balls (and k_{n+1} -balls) connecting the entrance and exit vertices in these balls.

Recall that the graphs $G(\mathcal{N}_n, p_n)$ in subsection 3.2 are complete for $b > 2K$ and $n \geq n_0$ (Lemma 3.3). We now compute a lower bound for

$$(4.2) \quad r_n := P(\text{a good } k_n\text{-ball in } B_{k_{n+1}}(\mathbf{0}) \text{ is connected to a good } k_{n+1}\text{-ball in } A_{n+1})$$

for large n . We have for large n , from equation (3.2),

$$\begin{aligned} r_n &\gtrsim 1 - \left(1 - \frac{a \log n \cdot N^{b \log n}}{N^{2k_{n+2}}}\right)^{\varepsilon^2 N^{k_n + k_{n+1}}} \\ &\sim 1 - \exp(-a \varepsilon^2 \log n \cdot N^{b \log n - [2(k_{n+2}) - k_n - k_{n+1}]}), \end{aligned}$$

and using (3.3)

$$2k_{n+2} - k_n - k_{n+1} = k_{n+1} - k_n + 2(k_{n+2} - k_{n+1}) \sim 3K \log n,$$

hence

$$(4.3) \quad r_n \gtrsim 1 - \exp(-a \varepsilon^2 \log n \cdot n^{(b-3K) \log N}).$$

There are $N^{K \log n}$ k_n -balls in $B_{k_{n+1}}(\mathbf{0})$, and $(N^{K \log(n+1)} - 1) \sim N^{K \log(n+1)}$ k_{n+1} -balls in A_{n+1} , and

$$\sum_n n^{K \log N} (n+1)^{K \log N} \exp(-a \varepsilon^2 \log n \cdot n^{(b-3K) \log N}) < \infty \quad \text{if } b > 3K,$$

which we now assume, so, for such b and for almost every realization of the percolation cluster there exists n_0 such that for all $n \geq n_0 \geq n_{00}$,

$$(4.4) \quad \text{every good } k_n\text{-ball in } B_{k_{n+1}}(\mathbf{0}) \text{ is connected in } \mathcal{C} \text{ to every good } k_{n+1}\text{-ball in } A_{n+1},$$

and this will be used in the construction below.

By the Rayleigh monotonicity principle, to prove that the random walk is transient on \mathcal{C} it suffices to show that it is transient on a subgraph of \mathcal{C} . Given a realization of the cluster and associated n_0 satisfying (4.4), we will construct a subgraph and a unit flow on it to satisfy the energy criterion for transience.

The flow has the following properties: Start with $B_{k_{n_0}}(\mathbf{0})$ and assume that $\mathbf{0}$ belongs to \mathcal{C} . Choose one edge from $B_{k_{n_0}}(\mathbf{0})$ to each one of the $N^{K \log(n_0+1)}$ k_{n_0} -balls in A_{n_0} . The unit flow entering at $\mathbf{0}$ is divided into $N^{K \log(n_0+1)}$ equal parts going to each one of the k_{n_0+1} -balls in A_{n_0} . The ball $B_{k_{n_0}}(\mathbf{0})$ has an internal structure which is the set of points (vertices) of Ω_N and edges that are contained in $\mathcal{C} \cap B_{k_{n_0}}$. There are many ways that the flow can go through paths from $\mathbf{0}$ to the (at least 1 and at most $N^{K \log(n_0+1)}$) exit-vertices in the cluster of $B_{k_{n_0}}(\mathbf{0})$, splitting appropriately at branch vertices on the paths in order to achieve the division of the flow as stated. Denote by E_0 the energy of the flow on the subgraph of the cluster connecting $\mathbf{0}$ to the exit vertices of $B_{k_{n_0}}(\mathbf{0})$. The flow will then pass through a series of disjoint subsets of edges in \mathcal{C} denoted $\{G_n\}_{n \geq 1}$ with the energies denoted by $\{E_n\}_{n \geq 1}$. G_1 consists of edges from the at most $N^{K \log(n_0+1)}$ exit-vertices in the cluster of $B_{k_{n_0}}(\mathbf{0})$ to the $N^{K \log(n_0+1)}$ disjoint k_{n_0} -balls in A_{n_0} and the edges connecting the entrance vertices in these balls to the exit vertices. Similarly for $n \geq 2$ G_n consists of edges from the $N^{K \log n}$ disjoint k_{n-1} -balls in A_{n-1} to the $N^{K \log(n+1)}$ disjoint k_n -balls in A_n and the edges connecting the (at most 2) entrance vertices in these balls to the (at most 3) exit vertices.

We now specify in detail the choice of the edges in G_n and the flow in each of these edges. For $n > n_0$ each of the $N^{K \log(n+1)}$ k_n -balls in A_n gets $1/N^{K \log(n+1)}$ amount of flow entering through 1 or 2 edges, which then goes through the internal structure of each k_{n+1} -ball and is then divided along 1,

2 or 3 edges to the $N^{K \log(n+2)}$ k_{n+2} -balls in A_{n+1} . To do this we first enumerate the $N^{K \log(n+1)}$ k_n -balls, in A_n denoted $B_1^n, \dots, B_{N^{K \log(n+1)}}^n$ and then enumerate the $N^{K \log(n+2)}$ k_{n+1} -balls in A_{n+1} denoted $B_1^{n+1}, \dots, B_{N^{K \log(n+2)}}^{n+1}$. We first choose edges between B_1^n and B_1^{n+1}, B_2^{n+1} and assign flow $1/N^{K \log(n+2)}$ to the edge to B_1^{n+1} and $1/N^{K \log(n+1)} - 1/N^{K \log(n+2)}$ to the edge to B_2^{n+1} . We then fill B_2^{n+1} up to level $1/N^{K \log(n+2)}$ from B_2^n and successively assign flows to edges from the B_i^n 's to the B_j^{n+1} 's so that each B_i^n becomes empty and each B_j^{n+1} is filled up by the end of the procedure. This can be done in several ways so that all the entrance flows and exit flows have the same order of magnitude.

This procedure is repeated for the successive A_n . This means that for each $n > n_0$, each one of the k_n -balls in A_n gets $1/N^{K \log(n+1)}$ total amount of entrance flow. Noting that for large n

$$\frac{1}{N^{K \log(n+1)}} < \frac{2}{N^{K \log(n+2)}},$$

each of the k_n -balls has 1 or 2 entrance edges and 1, 2 or 3 exit edges. Any entrance-exit pair in the k_n -balls can be connected by a path (within the ball by completeness) of expected length bounded by $CD(n_0)n$ (Lemma 3.5). Therefore each of the edges belongs to at most 6 paths and the expected energy of the flow (recall Definition 4.2) is then bounded by

$$(4.5) \quad 6 \sum_{n=n_0}^{\infty} N^{K \log(n+1)} \frac{CD(n_0)n}{N^{2K \log(n+1)}} = 6 \sum_{n=n_0}^{\infty} \frac{CD(n_0)n}{(n+1)^{K \log N}},$$

where the n th summand refers to the expected energy of the flow from entrance vertices in A_n to the entrance vertices in A_{n+1} . Then the expected energy of the flow is finite if

$$\sum_n \frac{n}{n^{K \log N}} < \infty,$$

which holds because $K > 2/\log N$. Hence with the assumption that $b > 3K$ we can construct a flow on a subgraph of \mathcal{C} with finite energy for almost every realization of the percolation cluster and therefore the random walk on the cluster is transient. Since this holds for $K \log N > 2$ and $b > 3K$, and we have assumed that $\alpha > b \log N$, then $\alpha > 6$ suffices. ■

Finally, we can give the main result.

Theorem 4.6 *Consider the simple random walk on the percolation cluster with $\delta = 1$, $C_2 > 0$. Then for almost every realization of the percolation cluster there exists a critical $\alpha_c \in (0, \infty)$ such that for $\alpha < \alpha_c$ the random walk is recurrent and for $\alpha > \alpha_c$ the random walk is transient.*

Proof. By Theorems 4.3 and 4.5 there exist $0 < \alpha_1 < \alpha_2 < \infty$ such that the random walk on the percolation cluster is recurrent for $\alpha \leq \alpha_1$ and transient for $\alpha = \alpha_2$. Moreover, given $\alpha < \alpha'$ we can construct the two associated percolation clusters (using the same C_2) on one probability space so that the α -cluster is a subgraph of the α' -cluster with probability one (see Remark 2.2 in [22]). But then the Rayleigh monotonicity principle implies that if the random walk on the α -cluster is transient it is also transient on the α' -cluster. We define $\alpha_c = \inf\{\alpha : \text{the walk on the } \alpha\text{-cluster is transient}\}$, which yields the desired result. ■

5 Further questions

Questions that could be addressed which lie outside the scope of the present paper are as follows.

We have focussed on connection probabilities with c_k having logarithmic and polynomial growth. It would be interesting to study existence of percolation with intermediate growth, and related questions for random walks.

What is the exact value of the critical α_c , and is the walk recurrent or transient at $\alpha = \alpha_c$?

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